RCT - REVISTA DE CIÊNCIA E TECNOLOGIA
www.revista.ufrr.br/rct

Relative Positions Between the Hyperplane and the n-Sphere

Joselito de Oliveira [1] and Wender Ferreira Lamounier [2]

[1]Departamento de Matemática - Universidade Federal de Roraima (UFRR)
Boa Vista – RR – Brazil

[2]Escola de Aplicação – Universidade Federal de Roraima (UFRR)
Boa Vista – RR – Brazil

joselito.oliveira@ufrr.br, wender.lamounier@ufrr.br

Abstract. This paper discusses are some topics Analytic Geometry, studied in basic education in the context of Euclidean space n-dimensional. Presents itself for example, the concepts of hyperplane and $(n-1)$-sphere, which correspond to the high school to the circle and line, respectively. And in the said geometry are studied the relative positions between straight line and circumference. Similarly, we study the relative positions between the hyperplane and the $(n-1)$-sphere in this space. In this context, it presents a theorem that characterizes the relative positions.
Keywords: Hyperplane. n-Sphere. Euclidean space.

1. Introduction

In the Plane Analytic Geometry we study the relative positions between circumference and straight line. In this work we present a study in Euclidean space $\mathbb{R}^n$ of the relative positions between the hyperplane and the $(n-1)$-sphere. Good part this paper is part of the dissertation of the second author, produced at Professional Master's degree in Mathematics in National Network - PROFMAT, Federal University of Roraima-UFRR, see reference [Lamounier 2014].

Initially, in section two, is introduced the mathematics necessary for the development of the study. In section three is studied the distance between point and hyperplane and are reminded the particular cases in Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. In section four is studied in Euclidean space $\mathbb{R}^n$, the relative positions between the hyperplane and the $(n-1)$-sphere. On that occasion, we noted that the study of the relative positions between hyperplanes and of the relative positions between $(n-1)$-spheres are in [Oliveira e Lamounier 2015].

2. Preliminaries

In this section, we will present the mathematical that will serve as a theoretical support for the development of the article. The demonstrations of results can be found in the references [Lima 2005] and [Spivak 2003, p. 6].

Definition 2.1. Let $n \in \mathbb{Z}_+$ be, denotes by $\mathbb{R}^n$ the cartesian product of n factors equal to $\mathbb{R}$, i.e., $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$.

The following operations make $\mathbb{R}^n$ a $\mathbb{R}$-vector space.

Given $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ vectors of $\mathbb{R}^n$ and a real number α, the sum operation $x + y$ and the operation product of a vector by a scalar $\alpha \cdot x$ are defined by:

i) $x + y = (x_1 + y_1, \ldots, x_n + y_n)$;
ii) $\alpha \cdot x = (\alpha \cdot x_1, \ldots, \alpha \cdot x_n)$.

Note 2.1. The neutral element of addition is the $0 = (0, \ldots, 0)$ and the symmetrical element of $x = (x_1, \ldots, x_n)$ is $-x = (-x_1, \ldots, -x_n)$, since $x + (-x) = 0$.

The concept of symmetry is as follows: be $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, y is the symmetrical of x if and only if $y_1 = -x_1, \ldots, y_n = -x_n$.

Given two vectors belonging to the vector space $\mathbb{R}^n$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the inner product of x and y considered here is given by

$$\langle x, y \rangle = \textstyle\sum_{i=1}^{n} x_i y_i,$$

known as usual inner product.

The vector space $\mathbb{R}^n$ on $\mathbb{R}$ with the usual inner product is called n-Euclidean space.

It is proved that the usual inner product satisfies the following properties:

1. $< x, x > \geq 0$, for all $x \in \mathbb{R}^n$ and $< x, y >= 0$ if and only if $x = 0$.
2. $< x, y >=< y, x >$, for all $x, y \in \mathbb{R}^n$.
3. $< x + y, z >=< x, z > + < y, z >$, for all $x, y, z \in \mathbb{R}^n$.
4. $< \lambda x, y >= \lambda < x, y >$, for all $x, y \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$.

Example 2.1. We have the following Euclidean spaces: the line $\mathbb{R}^1 = \mathbb{R}$; the plane $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ and the space $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$.

We remember that two vectors $x, y \in \mathbb{R}^n$ are orthogonal when $\langle x, y \rangle = 0$.

For our study we consider the Euclidean norm, that is, the real number given by $\|x\| = \sqrt{\langle x, x \rangle}$, where $x \in \mathbb{R}^n$.

From the inner product it is proved that the norm satisfies the following properties:

1. $\|x\| \geq 0$, for all $x \in \mathbb{R}$, and $\|x\| > 0$ if $x \neq 0$.
2. $\|\lambda x\| = |\lambda| \|x\|$, for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$.
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in \mathbb{R}^n$.

Based on the norm can define the distance $\mathbb{R}^n$ as follows:

Let $x, y \in \mathbb{R}^n$ be, the distance of x to y is defined by

$$d(x, y) = \|x - y\| \cdot$$

Let $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ be, points of $\mathbb{R}^n$, then

$$d(P, Q) = \sqrt{\textstyle\sum_{i=1}^{n} (p_i - q_i)^2}.$$

The concept of hyperplane can be found in [Coelho 2001] and [Lang 2003], but here will state considering the vector space $\mathbb{R}^n$.

Definition 2.2. Let v be a not null vector and P a point in Euclidean space $\mathbb{R}^n$. Denominates hyperplane to set

$$\Gamma_v^{n-1} = \{X \in \mathbb{R}^n \mid \langle X - P, v\rangle = 0\}.$$

Considering the point $P = (p_1, \ldots, p_n)$ and the not null vector $v = (v_1, \ldots, v_n)$, given $X = (x_1, \ldots, x_n) \in \Gamma_v^{n-1}$ then $\langle (X - P), v\rangle = 0$. Therefore $v \perp (X - P)$, where $X - P = (x_1 - p_1, \ldots, x_n - p_n)$. So,

$$\langle (x_1 - p_1, \ldots, x_n - p_n), (v_1, \ldots, v_n)\rangle = v_1x_1 + \ldots + v_nx_n - p_1v_1 - \ldots - p_nv_n.$$

That is,

$$v_1x_1 + \ldots + v_nx_n + d = 0, \text{ where } d = -p_1v_1 - \ldots - p_nv_n.$$

And this is the equation of the hyperplane Γ_v^{n-1} passing through point P and it is normal to vector $v = (v_1, \ldots, v_n)$.

Example 2.2. In the plane, case in which $n = 2$, the hyperplane is the well known equation of a line $v_1x_1 + v_2x_2 + d = 0$, object of study of plane analytic geometry.

Example 2.3. In the space, case in which $n = 3$, the hyperplane is the well known equation of a plane $v_1x_1 + v_2x_2 + v_3x_3 + d = 0$.

3. Distance from a point to a hyperplane

We start the session with the version of the Pythagorean theorem for Euclidean space $\mathbb{R}^n$, present in [Hönig 1976, p. 190] and proposed as an exercise in [Spivak 2003, p. 6]. Then, we present the definition of distance from a point to a hyperplane in the space $\mathbb{R}^n$.

Theorem 3.1 (Pythagoras theorem). Let A, B and C be points in $\mathbb{R}^n$ such that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$. Then,

$$\overrightarrow{AB} \perp \overrightarrow{BC} \iff \|\overrightarrow{AC}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2.$$

Proof. Let A, B and C be points in $\mathbb{R}^n$, then

$$\begin{aligned}
\|\overrightarrow{AC}\|^2 &= \|\overrightarrow{AB} + \overrightarrow{BC}\|^2 \\
&= \langle \overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{AB} + \overrightarrow{BC}\rangle \\
&= \langle \overrightarrow{AB}, \overrightarrow{AB}\rangle + 2\langle \overrightarrow{AB}, \overrightarrow{BC}\rangle + \langle \overrightarrow{BC}, \overrightarrow{BC}\rangle \\
&= \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2 + 2\langle \overrightarrow{AB}, \overrightarrow{BC}\rangle.
\end{aligned}$$

Therefore, $\overrightarrow{AB} \perp \overrightarrow{BC} \iff \|\overrightarrow{AC}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2$. □

Corollary 3.1. Let A, B and C be points in $\mathbb{R}^n$, such that $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{AB} \perp \overrightarrow{BC}$. Then:

i) $\|\overrightarrow{AB}\| \leq \|\overrightarrow{AC}\|$;
ii) $\|\overrightarrow{BC}\| \leq \|\overrightarrow{AC}\|$.

Proof. If $\overrightarrow{AB} = \overrightarrow{0}$ then, by Theorem 3.1, $\|\overrightarrow{AC}\|^2 = \|\overrightarrow{BC}\|^2 \Rightarrow \|\overrightarrow{AC}\| = \|\overrightarrow{BC}\|$.

Analogously, if $\overrightarrow{BC} = \overrightarrow{0}$ then, by Therem 3.1, we have $\|\overrightarrow{AC}\| = \|\overrightarrow{AB}\|$.

We will see now the case in which the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ not are null.

If $\overrightarrow{AB} \perp \overrightarrow{BC}$ by Theorem 3.1 we have

$$\|\overrightarrow{AC}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2 \Rightarrow \|\overrightarrow{AC}\|^2 > \|\overrightarrow{AB}\|^2 \Rightarrow \|\overrightarrow{AC}\| > \|\overrightarrow{AB}\|.$$

The proof of (ii) is analogous. □

Corollary 3.2. Let A, B and C be points in $\mathbb{R}^n$ such that $C \neq B$, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ e $\overrightarrow{AB} \perp \overrightarrow{BC}$. Then $d(A, B) < d(A, C)$.

Proof. By Corollary 3.1(i) and by Theorem 3.1 we obtain $d(A, B) < d(A, C)$. □

Definition 3.1. The distance from a point P_0 in $\mathbb{R}^n$ to a hyperplane Γ_v^{n-1}, where $P_0 \notin \Gamma_v^{n-1}$ is given as the lowest of the distances from P_0 to the points of Γ_v^{n-1}, that is,

$$d(P_0, \Gamma_v^{n-1}) = min\{d(P_0, Q); Q \in \Gamma_v^{n-1}\}.$$

The next statement will be considered here as an axiom, however, in the reference [Kreyszig 1978, p. 144] is a theorem that needs analysis techniques to perform the proof.

Axiom 3.1. Let Γ_v^{n-1} be a hyperplane, P_0 a point of the $\mathbb{R}^n$ and v a normal vector to Γ_v^{n-1}. There exists a unique point $P' \in \Gamma_v^{n-1}$ such that $\overrightarrow{P_0P'} // v$ and $d(P_0, \Gamma_v^{n-1}) = d(P_0, P')$.

Example 3.1.

a) As we know that the hyperplane in $\mathbb{R}^2$ is a line, we will denote by r. If P_0 is a point of the $\mathbb{R}^2 - r$, then by Axiom 3.1 there exists a unique point $Q \in r$ such that $d(P_0, r) = d(P_0, Q)$, where $\overrightarrow{P_0Q} \perp r$.
b) The hyperplane in $\mathbb{R}^3$ is a plane of the space and we will denote by π. If P_0 is a point of the $\mathbb{R}^3 - \pi$, then by Axiom 3.1 there exists a unique point $Q \in \pi$ such that $d(P_0, \pi) = d(P_0, Q)$, where $\overrightarrow{P_0Q} \perp \pi$.

Now we will present a formula for calculating the distance from a point to the hyperplane. Initially we will remember the formula in the following cases:

1. In $\mathbb{R}^2$, consider tha line $\Gamma_v^1 : ax + by + d = 0$ (hyperplane), generated by vector $v = (a, b)$, and the point $P_0 = (x_1^0, x_2^0) \in \mathbb{R}^2 - \Gamma_v^1$. The distance from point to plane is given by

$$d(P_0, \Gamma_v^1) = \frac{|ax_1^0 + bx_2^0 + d|}{\sqrt{a^2 + b^2}}.$$

2. In $\mathbb{R}^3$, consider the plane $\Gamma_v^2 : ax + by + cz + d = 0$ (hyperplane), generated by vector $v = (a, b, c)$, and the point $P_0 = (x_1^0, x_2^2, x_3^0) \in \mathbb{R}^3 - \Gamma_v^2$. The distance from the point to plane is given by

$$d(P_0, \Gamma_v^2) = \frac{|ax_1^0 + bx_2^0 + cx_3^0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

The following theorem gives us the formula for calculating the distance from a point $P_0 \in \mathbb{R}^n$ to a hyperplane Γ_v^{n-1}. We observed that it is not in [Lamounier 2014].

Theorem 3.2. let $\Gamma_v^{n-1} : a_1x_1 + \cdots + a_nx_n + d = 0$ a hyperplane and let $P_0 = (x_1^0, \cdot \cdots, x_n^0)$ be a point in $\mathbb{R}^n - \Gamma_v^{n-1}$. Then a distance between the point P_0 and the hyperplane Γ_v^{n-1} is given by:

$$d(P_0, \Gamma_v^{n-1}) = \frac{|a_1x_1^0 + \cdots + a_nx_n^0 + d|}{\sqrt{a_1^2 + \cdots + a_n^2}}.$$

Proof. The proof was adapted from [Steinbruch e Winterle 1987, p.196].

Let $\Gamma_v^{n-1} : a_1x_1 + \cdots + a_nx_n + d = 0$ be the hyperplane equation with the normal vector not null $v = (a_1, \cdots, a_n)$, through a point P and $P_0 = (x_1^0, \cdot \cdots, x_n^0) \in \mathbb{R}^n - \Gamma_v^{n-1}$, then by Axiom 3.1 exists a unique point $Q \in \Gamma_v^{n-1}$ such that $d(P_0, \Gamma_v^{n-1}) = d(P_0, Q)$, where $\overrightarrow{P_0Q} // v$. Then, exist $\lambda \in \mathbb{R}$ such that $Q - P_0 = \lambda v$, that is,

$$P_0 = Q - \lambda v. \tag{1}$$

Being $P_0 = Q - cv$ then $P_0 - P = (Q - P) + \lambda v$. Therefore,

$$\begin{aligned} < P_0 - P, v > &= < Q - P, v > + \lambda < v, v > \\ &= \lambda < v, v > . \end{aligned}$$

Being $v \neq 0$, then

$$\lambda = \frac{< P_0 - P, v >}{\| v \|^2}. \tag{2}$$

From (1) and (2) we obtain

$$P_0 = Q - \frac{< P_0 - P, v >}{\| v \|^2} v.$$

Then, the distance from P_0 to closest point Q is equal to

$$\begin{aligned} d(P_0, Q) &= \| P_0 - Q \| \\ &= \| - \frac{< P_0 - P, v >}{\| v \|^2} v \| \\ &= \frac{|< P_0 - P, v >|}{\| v \|}. \end{aligned}$$

And as $d(\Gamma_v^{n-1}, P_0) = d(P_0, Q)$ and $d = - < P, v >$, then

$$d(P_0, \Gamma_v^{n-1}) = \frac{|a_1 x_1^0 + \ldots + a_n x_n^0 + d|}{\sqrt{a_1^2 + \ldots + a_n^2}}.$$

□

We present below the definition of $(n-1)$-sphere, or hypersphere, and the equation that describes it in $\mathbb{R}^n$. In [Millman 1977] and [Mendelson 1990] the definition of $(n-1)$-sphere is restricted to the case where the radius worth one and the center is the origin. Without loss of generality we will consider the $(n-1)$-sphere in which the radius is greater than or equal to one and with center at a point arbitrary of $\mathbb{R}^n$.

Definition 3.2. A $(n-1)$-sphere in $\mathbb{R}^n$ with radius $r > 0$ and center c is the set

$$\mathbb{S}_r^{n-1}(c) = \{x \in \mathbb{R}^n; d(c, x) = r\}, \text{ where } n \text{ is a positive integer.}$$

Note that, being $d(c, x) = r$, where $x = (x_1, ..., x_n)$ and $c = (c_1, ..., c_n)$, then $(x_1 - c_1)^2 + ... + (x_n - c_n)^2 = r^2$ is the equation of the $(n-1)$-sphere of center c and radius r. Therefore, the $(n-1)$-sphere can be written as the set

$$\mathbb{S}_r^{n-1}(c) = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} (x_i - c_i)^2 = r^2\}.$$

Example 3.2. Given $c = (c_1, ..., c_n) \in \mathbb{R}^n$ e $r > 0$, then:

1. For $n = 1$, the 0-sphere $\mathbb{S}_r^0(c) = \{x \in \mathbb{R} \mid (x - c)^2 = r^2\} = \{c - r, c + r\}$;
2. For $n = 2$, the 1-sphere

$$\mathbb{S}_r^1(c) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \sum_{i=1}^{2} (x_i - c_i)^2 = r^2\},$$

   is the circle with center $c = (c_1, c_2)$ and radius $r > 0$.
3. For $n = 3$, the 2-sphere

$$\mathbb{S}_r^2(c) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^{3} (x_i - c_i)^2 = r^2\},$$

   is the sphere with center $c = (c_1, c_2, c_3)$ and radius $r > 0$.

4. Relative positions between hyperplane and $(n-1)$-sphere

In the basic education we study the relative positions between the line and the circle, in this case the line, as we saw earlier, is a hyperplane and the circle is a 1-sphere in $\mathbb{R}^2$. In this section we study these relative positions in space Euclidean $\mathbb{R}^n$.

The next theorem characterizes the relative position between a hyperplane and $(n-1)$ - sphere. Cited in the article [Oliveira e Lamounier 2015], without his proof, it generalizes the cases where $n = 2$ (plane) and $n = 3$ (space). Here, we present a demonstration of the said result.

Theorem 4.1. Let Γ_v^{n-1} be a hyperplane passing through point $p = (p_1, ..., p_n)$ in $\mathbb{R}^n$ and $v = (v_1, ..., v_n)$. Let $\mathbb{S}_r^{n-1}(c)$ be a $(n-1)$-sphere with center $c = (c_1, ..., c_n)$ and radius $r > 0$. Then:

a) $d(c, \Gamma_v^{n-1}) > r$ if and only if $\Gamma_v^{n-1} \cap \mathbb{S}_r^{n-1}(c) = \varnothing$;

b) $d(c, \Gamma_v^{n-1}) = r$ if and only if $\Gamma_v^{n-1} \cap \mathbb{S}_r^{n-1}(c) = \{p_0\}$. In this case we say that the hyperplane is tangent to $(n-1)$ - sphere;

c) $d(c, \Gamma_v^{n-1}) < r$ if and only if $\mathbb{S}_r^{n-1}(c) \cap \Gamma_v^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(q)$, where $k = d(c, \Gamma_v^{n-1})$ and $\mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(q)$ is the $(n-2)$- sphere contained in the hyperplane Γ_v^{n-1} which has radius $\sqrt{r^2-k^2}$ and center q, point of intersection of the plane Γ_v^{n-1} with the line l normal to Γ_v^{n-1} passing through c, with normal vector v.

Proof. Without loss of generality we will consider:

i) The $(n-1)$-sphere $\mathbb{S}_r^{n-1}(c) = \{(x_1, ..., x_n) \in \mathbb{R}^n; x_1^2 + ... + x_n^2 = r^2\}$, with center at the origin $o = (0, ..., 0)$, wich of simplified mode is denoted by $\mathbb{S}_r^{n-1}$;

ii) The hyperplane $\Gamma_{e_n}^{n-1} = \{(x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n;\ x_n = k \text{ constante }\}$, where $e_n = (0, ..., 0, 1)$ is the normal vector to hyperplane.

In this first part of the proof we will prove the implication from left to right.

a1) Suppose that $d(o, \Gamma_{e_n}^{n-1}) > r$, then by Axiom 3.1, there exists a unique $p' \in \Gamma_{e_n}^{n-1}$ such that $\overrightarrow{op'} \perp \Gamma_{e_n}^{n-1}$ and $d(o, \Gamma_{e_n}^{n-1}) = d(o, p') > r$.
So we have:
$$d(o, q) \geq d(o, p') > r \text{ for all } q \in \Gamma_{e_n}^{n-1}.$$
Then $q \notin \mathbb{S}_r^{n-1}$ for all $q \in \Gamma_{e_n}^{n-1}$ and therefore $\mathbb{S}_r^{n-1} \cap \Gamma_{e_n}^{n-1} = \varnothing$.

b1) Suppose that $d(0, \Gamma_{e_n}^{n-1}) = r$, we conclude then, by Axiom 3.1, that exists a unique $q_0 \in \Gamma_{e_n}^{n-1}$ such that $\overrightarrow{oq_0} \perp \Gamma_{e_n}^{n-1}$ and $d(o, \Gamma_{e_n}^{n-1}) = d(o, q_0)$, then $q_0 \in \mathbb{S}_r^{n-1}$.
Let $b \in \Gamma_{e_n}^{n-1}$ be then $d(o, b) \geqslant d(o, q_0)$. Therefore, for all $b \in \Gamma^{n-1} - \{q_0\}$, we have $b \notin \mathbb{S}_r^{n-1}$. Therefore $\Gamma_{e_n}^{n-1} \cap \mathbb{S}_r^{n-1} = \{q_0\}$.

c1) If $d(o, \Gamma_{e_n}^{n-1}) < r$, then by Axiom 3.1 that exists a unique point $p_0 \in \Gamma_{e_n}^{n-1}$ such that $d(o, \Gamma_{e_n}^{n-1}) = d(p_0, o)$ where $\overrightarrow{op_0} \perp \Gamma_{e_n}^{n-1}$ and $p_0 \in l$, where l is a line generated by $e_n = (0, ..., 0, 1)$, therefore $p_0 = (0, ..., 0, k)$.
Being $d(o, \Gamma_{e_n}^{n-1}) < r$ then $d(o, p_0) =\mid k \mid < r$ and therefore $r^2 - k^2 > 0$.
We can consider $\mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0) = \{p \in \Gamma_{e_n}^{n-1}; d(p, p_0) = \sqrt{r^2-k^2}\}$.

Let $(x_1, ..., x_{n-1}, k) \in \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0)$ be, then

$$\begin{aligned} d(p, p_0) &= \sqrt{r^2-k^2} \\ \sqrt{(x_1-0)^2 + ... + (x_{n-1}-0)^2 + (k-k)^2} &= \sqrt{r^2-k^2} \\ x_1^2 + ... + x_{n-1}^2 &= r^2 - k^2 \\ x_1^2 + ... + x_{n-1}^2 + k^2 &= r^2. \end{aligned}$$

We concluded that $(x_1, ..., x_{n-1}, k) \in \mathbb{S}_r^{n-1}$. Therefore $\mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0) \subset \mathbb{S}_r^{n-1}$ and as $\mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0) \subset \Gamma^{n-1}_{e_n}$ then

$$\mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0) \subset \mathbb{S}_r^{n-1} \cap \Gamma^{n-1}_{e_n}. \tag{3}$$

We concluded then that $\mathbb{S}_r^{n-1} \cap \Gamma^{n-1}_{e_n} \neq \varnothing$. Now, let $p \in \mathbb{S}_r^{n-1} \cap \Gamma^{n-1}_{e_n}$ be, this is, $p \in \Gamma^{n-1}_{e_n}$ and $p \in \mathbb{S}_r^{n-1}$, then

$$\begin{aligned} x_1^2 + ... + x_{n-1}^2 + k^2 &= r^2 \\ x_1^2 + ... + x_{n-1}^2 &= r^2 - k^2. \end{aligned}$$

Then $p \in \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0)$ and therefore

$$\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} \subset \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0). \tag{4}$$

From (3) and (4) concluded that

$$\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_0).$$

In this second part of the proof we will prove the implication from rigth to left .

$a2)$ We will prove that: If $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \varnothing$ then $d(0, \Gamma^{n-1}) > r$.

Equivalently: if $d(o, \Gamma^{n-1}) \leq r$, then $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} \neq \varnothing$. But this follows from $(b1)$ e $(c1)$.
If $d(o, \Gamma^{n-1}_{e_n}) = r$ from $(b1)$ we have $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \{p\} \neq \varnothing$.
If $d(o, \Gamma^{n-1}_{e_n}) < r$ from $(c1)$ we have $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_o) \neq \varnothing$, where $k = d(o, \Gamma^{n-1}_{e_n})$.
Therefore, if $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \varnothing$, then $d(o, \Gamma^{n-1}_{e_n}) > r$.

$b2)$ We will prove that: If $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \{p_0\}$, then $d(o, \Gamma^{n-1}_{e_n}) = r$.

Equivalently, if $d(o, \Gamma^{n-1}_{e_n}) \neq r$ then $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1}$ it is not a single point.
For $d(o, \Gamma^{n-1}_{e_n}) \neq r$ we have two cases: $d(o, \Gamma^{n-1}_{e_n}) > r$ or $d(o, \Gamma^{n-1}_{e_n}) < r$.
In the case where $d(o, \Gamma^{n-1}_{e_n}) > r$, from $(a1)$ we have $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \varnothing$.
While, if $d(o, \Gamma^{n-1}_{e_n}) < r$, from $(c1)$ we have $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_o)$.
Therefore, if $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \{p_0\}$, then $d(o, \Gamma^{n-1}_{e_n}) = r$.

$c2)$ Now we will prove that: If $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_o)$ then $d(o, \Gamma^{n-1}_{e_n}) < r$.

Equivalently: if $d(o, \Gamma^{n-1}_{e_n}) \geq r$, then $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} \neq \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_o)$.
If $d(o, \Gamma^{n-1}_{e_n}) = r$ by proved in $(b1)$ we have that $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \{p_0\}$.
And if $d(o, \Gamma^{n-1}_{e_n}) > r$, from $(a1)$ we have $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \varnothing$.
Thefore, if $\Gamma^{n-1}_{e_n} \cap \mathbb{S}_r^{n-1} = \mathbb{S}^{n-2}_{\sqrt{r^2-k^2}}(p_o)$, then $d(o, \Gamma^{n-1}_{e_n}) < r$.

□

Let us now see examples of relative positions between plane Γ^2_v and unitary 2-sphere and center at the origin $\mathbb{S}^2_1$.

Example 4.1.

If $d(o, \Gamma_v^2) > 1$ then the plane Γ_v^2 does not intercept the 2-sphere unitary $\mathbb{S}_1^2$. Let see a example: We will consider the hyperplane Γ_v^2 with equation : $x - z + 2 = 0$, where $v = (1, 0, -1)$, then by Theorem 3.2 $d(o, \Gamma_v^2) = \sqrt{2} > 1$. By Theorem 4.1 we have $\mathbb{S}_1^2 \cap \Gamma_v^2 = \emptyset$. See the figure 4.1.

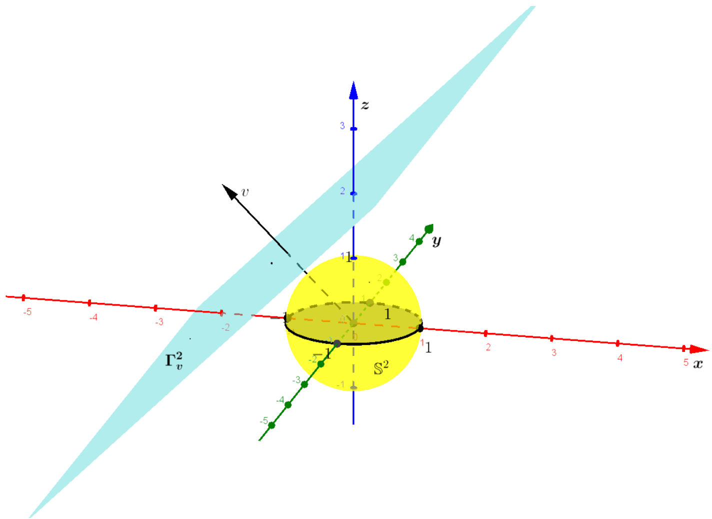

Figure 1.

Example 4.2.

If $d(o, \Gamma_v^2) = 1$ then the hyperplane Γ_v^2 is tangent to 2-sphere $\mathbb{S}_1^2$, in the point P. Let see the example: We will consider the hyperplane Γ_v^2 with equation $x - z + \sqrt{2} = 0$, where $v = (1, 0, -1)$, then by Theorem 3.2 we have $d(o, \Gamma_v^2) = 1$. By Theorem 4.1 we have $\mathbb{S}_1^2 \cap \Gamma_v^2 = \{P\}$, where $P = (-\dfrac{\sqrt{2}}{2}, 0, \dfrac{\sqrt{2}}{2})$. See the figure 4.2.

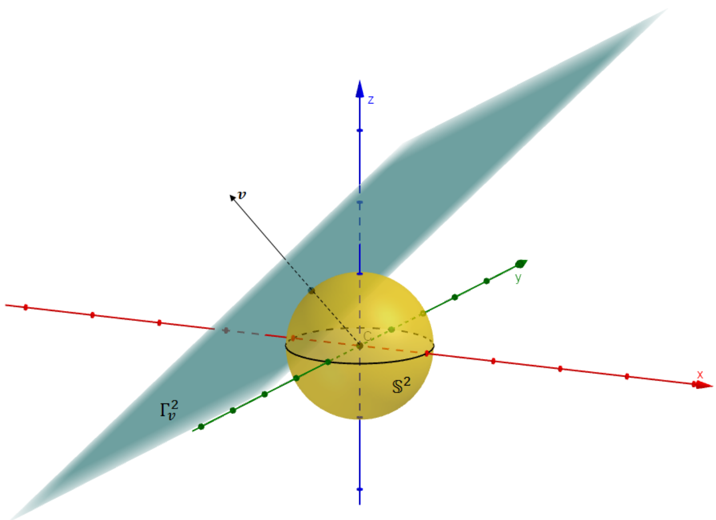

Figure 2.

Example 4.3.

If $d(o, \Gamma_v^2) < 1$ then the plane Γ_v^2 intersects 2-sphere $\mathbb{S}_1^2$ forming a 1-sphere $\mathbb{S}_r^1(q)$ (circumference) of radius $r = \sqrt{1 - d(c, \Gamma_v^2)^2}$, as shown the example : Consider the hyperplane Γ_v^2 with equation $y - z - \dfrac{\sqrt{2}}{2} = 0$, where $v = (0, 1, -1)$, then by Theorem 3.2 $d(o, \Gamma_v^2) = 1/2 < 1$. By Theorem 4.1 we have $\mathbb{S}_1^2 \cap \Gamma_v^2 = \mathbb{S}^1_{\sqrt{3}/2}(q)$, where $q = (0, \dfrac{\sqrt{2}}{4}, -\dfrac{\sqrt{2}}{4})$. See the figure 4.3.

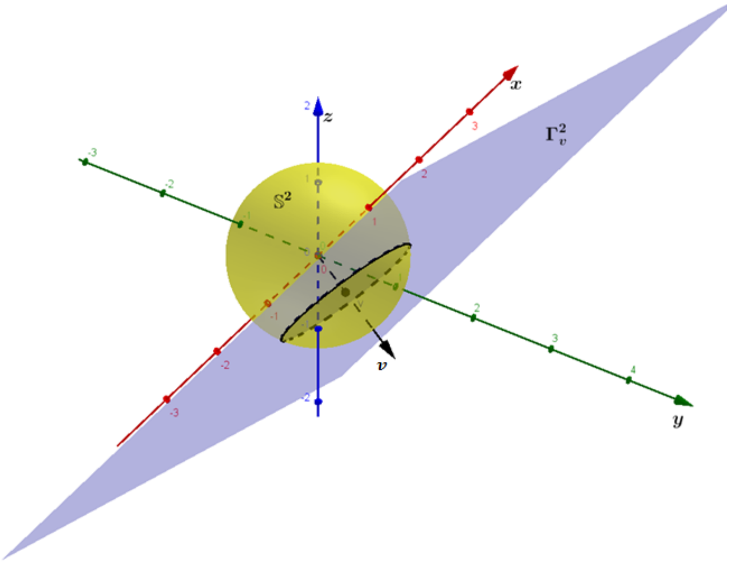

Figure 3.

5. Conclusions

In this article we studied the relative positions between the hyperplane and the n-sphere. Was presented, a result that generalizes to the Euclidean n-space, the problem of how to characterize the relative positions: between the line and the circle, in the case of plane, and between the plane and the sphere, in the case of Euclidean 3-space. We call attention to the case of the characterization of the relative position of the hyperplane with the (n-1)-esfera, when the distance to the hyperplane is less than the radius of the sphere, is a (n-2)-sphere and not a circle as in the case of the Euclidean 3-space.

References

Coelho, F. U.; Lourenço, M. L.(2001). Um curso de álgebra linear. São Paulo: Editora da Universidade de São Paulo.

Hönig, C. S. (1976). Aplicações da topologia à análise. Rio de Janeiro: IMPA.

Kreyszig, E. (1978). introductory functional analysis with applications. United States of America: Jonh Wiley & Sons.

Lamounier, W. F..(2014). A geometria análitica do ensino médio no contexto do espaço euclidiano $\mathbb{R}^n$. Boa Vista. 61f. Dissertação (Mestrado Profissional em Matemática em Rede Nacional-PROFMAT)- Universidade Federal de Roraima.

Lang, S.(2003). Álgebra linear: traduzido da terceira edição em inglês. 1. ed. Rio de Janeiro: Ciência Moderna.

Lima, E. L.(2005). Curso de análise Vol.2. 8. ed. Rio de Janeiro: IMPA.

Millman, R. S.; Porker, G. D.(1977). Elements of differential geometry New Jersey: Prentice Hall.

Mendelson, B.(1990). Introduction to topology New York: Dover Publications.

Oliveira, J.; Lamounier, W. F. (2015). Hiperplano e n-esfera: posições relativas. RCT-Revista de Ciência e Tecnologia. v.1.n.1.

Santos, N. M.(2007). Vetores e matrizes: uma introdução a álgebra linear. 4. ed. São Paulo: Thomson Pioneira.

Spivak, M.(2003). O cálculo em variedades. Rio de Janeiro: Editora Ciência Moderna Ltda.

Steinbruch, A.; Winterle, P.(1987). Geometria analítica. São Paulo: Pearson Makron Books.